The physics of electrokinetic devices applying and adapting the Child-Langmuir Law derivation for vacuum diodes

Part 1: electrokinetic devices in a vacuum

by Leon Tribe (leon.tribe@gmail.com)
16th January 2007

Recommended Reading

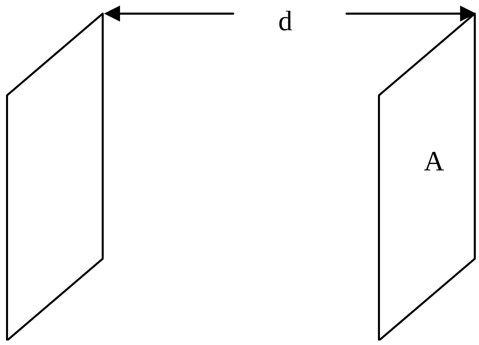
'Introduction to Electrodynamics' by David J. Griffiths. This and other related physics textbooks can be purchased here: Amazon Electrokinetic Physics Books Links

Introduction

The Child-Langmuir equation describes the characteristics of a parallel plate vacuum diode. By using this approach, we can derive a one dimensional expression for the characteristic properties of Lifters and related electrokinetic devices and by drawing an analogy to the vacuum diode, gain insights into the Lifter's subtle properties.

Derivation of the original Child-Langmuir equation

In a vacuum diode, let us have 2 plates, one plate grounded and the other at $V = V_0$. Let them be a distance 'd' apart and let us assume they are very large so that we can neglect end effects (Area = A >> d^2). Therefore, all properties will only depend on 'x' and we can make a 1-D approximation.



We know:

$$\frac{d^2V}{dx^2} = -\frac{\rho(x)}{\varepsilon_0} \text{ (Poisson's equation) (1)}$$

$$E = -\frac{dV}{dx} \text{ (Definition of potential) (2)}$$

$F = qE = ma$ (definition of electrostatic force and Newtonian force) (3)

Playing with (2) and (3):

$$F = -q\frac{dV}{dx} = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = mv\frac{dv}{dx} \text{ (4)}$$

Therefore taking (4), integrating both sides and substituting variables:

$$-q\frac{dV}{dx} = mv\frac{dv}{dx} \Rightarrow \frac{-q}{m}\int_0^{V(x)} dV = \int_0^{v(x)} vdv + \text{C}$$

Performing this integral we get:

$$\frac{-q}{m}V(x) = \frac{1}{2}v^2(x) + C$$

Assuming that at x=0, v=0, this becomes

$$\frac{-q}{m}V(x) = \frac{1}{2}v^2(x)$$

Rearranging, we obtain:

$$v(x) = \sqrt{\frac{-2q}{m}V(x)} \text{ (5)}$$

As:

$I = -jA = -\rho vA$ (Definition of current) (6)

Combining (5) and (6), we get:

$$I = -\rho(x)v(x)A = -\rho(x)A\sqrt{\frac{-2q}{m}V(x)} \text{ (7)}$$

Combining (1) and (7) we get:

$$\frac{d^2V}{dx^2} = \frac{1}{\varepsilon_0}\frac{I}{A}\sqrt{\frac{m}{-2qV(x)}} = \frac{1}{\varepsilon_0}\frac{I}{A}\sqrt{\frac{m}{-2q}}V(x)^{-\frac{1}{2}} \text{ (8)}$$

Multiplying both sides of (8) by $2\frac{dV}{dx}$ we get:

$$2\frac{dV}{dx}\frac{d^2V}{dx^2} = \frac{1}{\varepsilon_0}\frac{I}{A}\sqrt{\frac{-m}{2q}}2\frac{dV}{dx}V(x)^{-\frac{1}{2}} \Rightarrow \frac{d}{dx}\left(\frac{dV}{dx}\right)^2 = \frac{4}{\varepsilon_0}\frac{I}{A}\sqrt{\frac{-m}{2q}}\frac{d}{dx}\left(V^{1/2}(x)\right) \text{(9)}$$

Integrating both sides of (9) yields:

$$\left(\frac{dV}{dx}\right)^2 = \frac{4}{\varepsilon_0}\frac{I}{A}\sqrt{\frac{-m}{2q}}V^{1/2}(x) + C$$

At x = 0, V(x) = 0 and assuming $E(x) = -\frac{dV}{dx} = 0$ (verified experimentally and this is also vindicated when the solution is found, which has to be unique), we find that C = 0 giving:

$$\left(\frac{dV}{dx}\right)^2 = \frac{4}{\varepsilon_0}\frac{I}{A}\sqrt{\frac{-m}{2q}}V^{1/2}(x)\ (10)$$

Integrating both sides and again saying that at x = 0. V(x) = 0 and E(x) = 0

$$\left(\frac{dV}{dx}\right) = 2\left(\frac{1}{\varepsilon_0}\frac{I}{A}\right)^{1/2}\left(\frac{-m}{2q}\right)^{1/4} V^{1/4}(x)\ (11)$$

Inverting

$$\frac{dx}{dV} = \frac{1}{2}\left(\frac{1}{\varepsilon_0}\frac{I}{A}\right)^{-1/2}\left(\frac{-m}{2q}\right)^{-1/4} V^{-1/4}(\mathrm{x})\ (12)$$

Integrating by V(x) and saying at x = 0, V(x) = 0

$$x = \frac{1}{2}\left(\frac{1}{\varepsilon_0}\frac{I}{A}\right)^{-1/2}\left(\frac{-m}{2q}\right)^{-1/4}\frac{4}{3}V^{3/4}(x) = \frac{2}{3}\left(\frac{1}{\varepsilon_0}\frac{I}{A}\right)^{-1/2}\left(\frac{-m}{2q}\right)^{-1/4} V^{3/4}(x)\ (13)$$

Rearranging for V(x) yields

$$V(x) = \left(\frac{3}{2}\right)^{4/3}\left(\frac{1}{\varepsilon_0}\frac{I}{A}\right)^{2/3}\left(\frac{-m}{2q}\right)^{1/3} x^{4/3} = \left(\frac{9I}{4\varepsilon_0 A}\right)^{2/3}\left(\frac{-m}{2q}\right)^{1/3} x^{4/3}\ (14)$$

At x = d, $V = V_0$, therefore

$$V_0 = \left(\frac{9}{4\varepsilon_0 A}\right)^{2/3}\left(\frac{-m}{2q}\right)^{1/3} I^{2/3} d^{4/3}\ (15)$$

Rearranging for I

$$I = V_0^{3/2}\left(\frac{4}{9}\right)\varepsilon_0 A\left(\frac{-2q}{m}\right)^{1/2} d^{-2}\ (16)$$

This is the Child-Langmuir law. The numerical constant (in this case $\frac{4}{9}$) can change depending on the assumed shape of the anode and cathode (for us we assumed two planes). The power of this equation is in highlighting how the parameters scale with each other.

Combining (14) and (16) and writing in terms of V(x) yields

$$V(x) = V_0 d^{-4/3} x^{4/3}\ (17)$$

Combining (2) and (17) we can derive a form for E(x)

$$E(x) = -V_0 d^{-4/3}\frac{4}{3}x^{1/3}\ (18)$$

Combining (5) and (17) gives us

$$v(x) = \sqrt{\frac{-2q}{m}} V_0^{1/2} d^{-2/3} x^{2/3}\ (19)$$

Combining (1), (2) we get

$$-\frac{dE}{dx} = -\frac{d}{dx}\left(-\frac{dV}{dx}\right) = \frac{d^2V}{dx^2} = -\frac{\rho(x)}{\varepsilon_0} \quad (20)$$

Combining (18) and (20) gives us

$$\frac{dE}{dx} = -V_0 d^{-4/3}\frac{4}{9}x^{-2/3} = \frac{\rho(x)}{\varepsilon_0}$$

Therefore

$$\rho(x) = -V_0\varepsilon_0 d^{-4/3}\frac{4}{9}x^{-2/3} \quad (21)$$

Finally, dividing (3) by A gives us

$$\frac{F(x)}{A} = \rho(x)E(x) = V_0{}^2\varepsilon_0 d^{-8/3}\frac{16}{27}x^{-1/3} \quad (22)$$

Integrating the force over the gap and considering the force will be equal and opposite to the force on the device we obtain

$$F_T = \int_0^d F(x)dx = AV_0{}^2\varepsilon_0 d^{-8/3}\frac{8}{9}d^{2/3} = AV_0{}^2\varepsilon_0 d^{-2}\frac{8}{9} \quad (23)$$

NB: F(x) is effectively a force per unit length in the x-direction, apologies for the unit confusion that may result from my naming conventions.

Using (15), in terms of current we have

$$F_T = A\varepsilon_0 d^{-2}\frac{8}{9}\left(\frac{9}{4\varepsilon_0 A}\right)^{4/3}\left(\frac{-m}{2q}\right)^{2/3} I^{4/3}d^{8/3} = (A\varepsilon_0)^{-1/3}d^{2/3}\frac{243}{32}\left(\frac{-m}{2q}\right)^{2/3} I^{4/3} \quad (24)$$

Consequences for electrokinetic devices

There are two potential sources for any observed movement seen by an electrokinetic device in a vacuum. The first is a simple momentum consideration (mass-driven movement). The second is the force caused by the attraction of the electrons in the gap to the induced charge on the electrodes (charge-driven force).

However, assuming every electron that is emitted from one electrode is collected at the other electrode, while conservation of momentum holds and Newton’s third law is valid, all gains made in the gap will be lost on collection. In other words, while (3) holds, the attractive force of the electrons in the gap pulling the device one way will be cancelled out by the impact of the electrons as they are collected coming the other way.

Let us now assume we conduct an experiment in a vacuum chamber and observe a force. Can we account for this? In fact we can if we relax one of our assumptions. Let us assume that not every electron emitted is collected but rather hits the wall of our vacuum chamber. Rather than its momentum cancelling the force experienced by the device, it is lost to the chamber walls. This breaks (3) unless we also consider the change in momentum of the chamber walls due to the impact of the electrons (in practice of course this would not be noticeable and would wash out with friction).

So how do we predict our new-found force? Well not easily but we will get an indication from the current loss. If we measure the current at the emitting electrode and at the collecting electrode, we can determine our current loss. In (24) we have an expression for the force in terms of the current. The loss in current will directly correlate to the force lost to the chamber or equivalently, the force gained by the device. Why do I say an indication? Simply because our simple 1-D model does not consider the walls of the chamber, its geometry or any other aspect of its existence.

In Part 2, I will adapt this for an electrokinetic device in air.